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Dynamics of Stochastic Systems with Memory (Mathematics and Statistics Colloquium, Wright State University)

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DYNAMICS
OF
STOCHASTIC SYSTEMS
WITH MEMORY

Dayton, Ohio : January 7, 2000

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Deterministic ODE's: Stable Manifolds

ODE on \mathbf{R}^d :

$$dx(t) = h(x(t)) dt \quad (ODE)$$

driven by a vector field $h : \mathbf{R}^d \rightarrow \mathbf{R}^d$, C_b^k ; viz. all derivatives $D^j h$, $1 \leq j \leq k$, continuous and globally bounded.

Assume *hyperbolic equilibrium* at 0: $h(0) = 0$; $Dh(0) \in L(\mathbf{R}^d)$ has all eigenvalues off imaginary axis.

Then (ODE) has a C_b^k flow $\phi : \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ s.t.

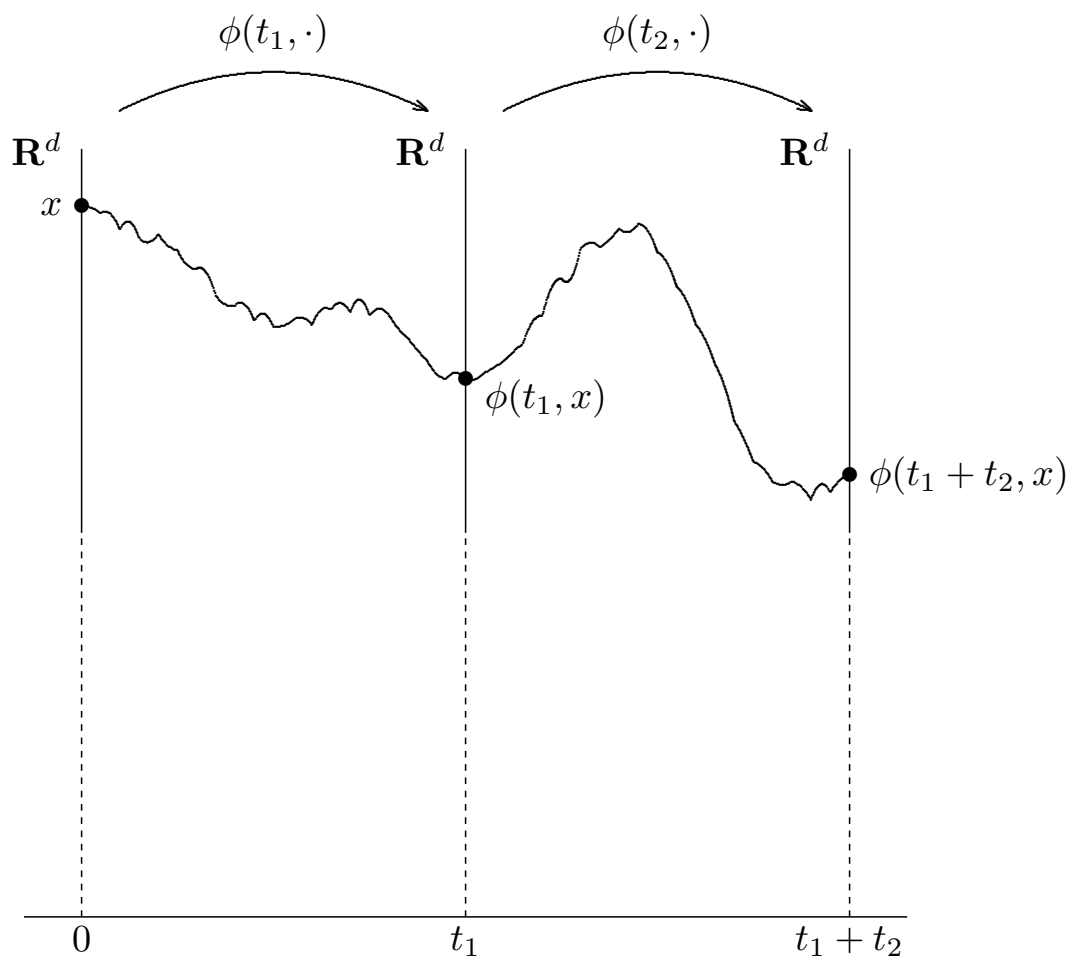
- (i) $\phi(\cdot, x)$ = unique solution of (ODE) through $x \in \mathbf{R}^d$.
- (ii) $\phi(t, 0) = 0, t \in \mathbf{R}$.
- (iii) Group property:

$$\phi(t_1 + t_2, \cdot) = \phi(t_2, \cdot) \circ \phi(t_1, \cdot), \quad t_1, t_2 \in \mathbf{R}$$

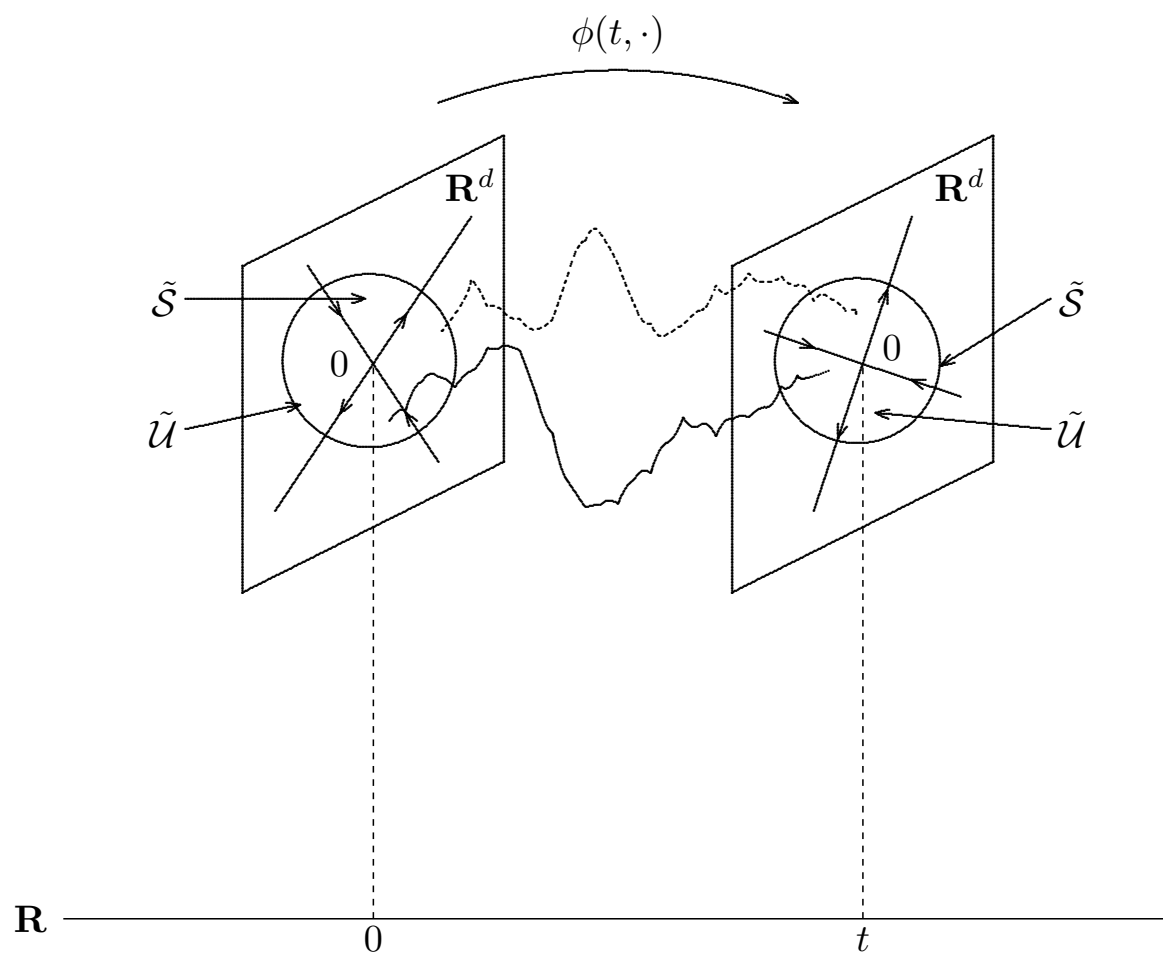
- (iv) Local flow-invariant stable/unstable C^k manifolds in a neighborhood of 0.

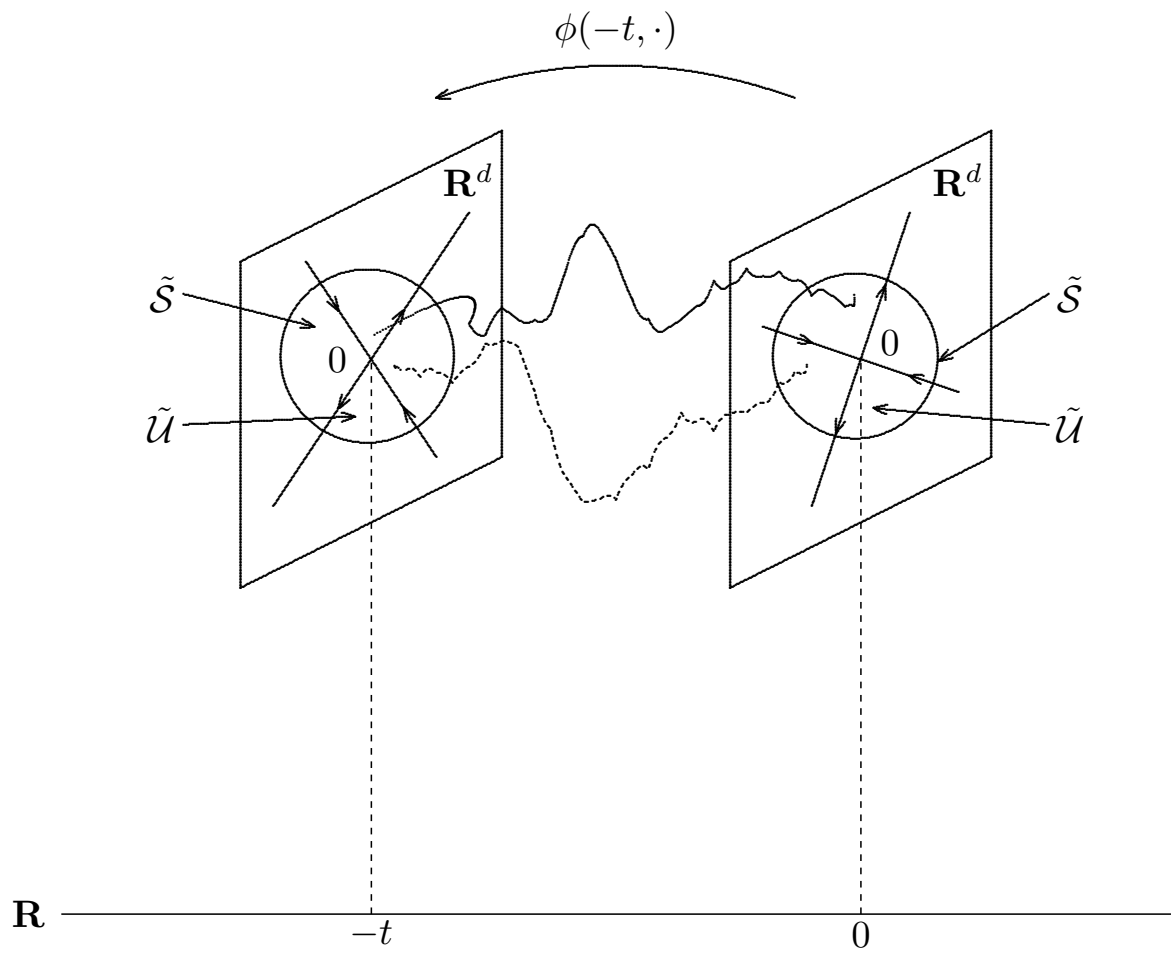
Properties (i)-(iv) are “generic” among all vector fields.

The Flow



Local Stable/Unstable Manifolds





What happens
if vector field
is noisy??

CASE STUDY:

Stochastic Systems with Memory

Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.
- *Local Stable Manifold Theorem* for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proof: Ruelle-Oseledec multiplicative ergodic theory + perfection techniques.

The Cocycle

$(\Omega, \mathcal{F}, P) :=$ complete probability space.

$\theta : \mathbf{R}^+ \times \Omega \rightarrow \Omega$ a P -preserving (ergodic) semigroup on (Ω, \mathcal{F}, P) .

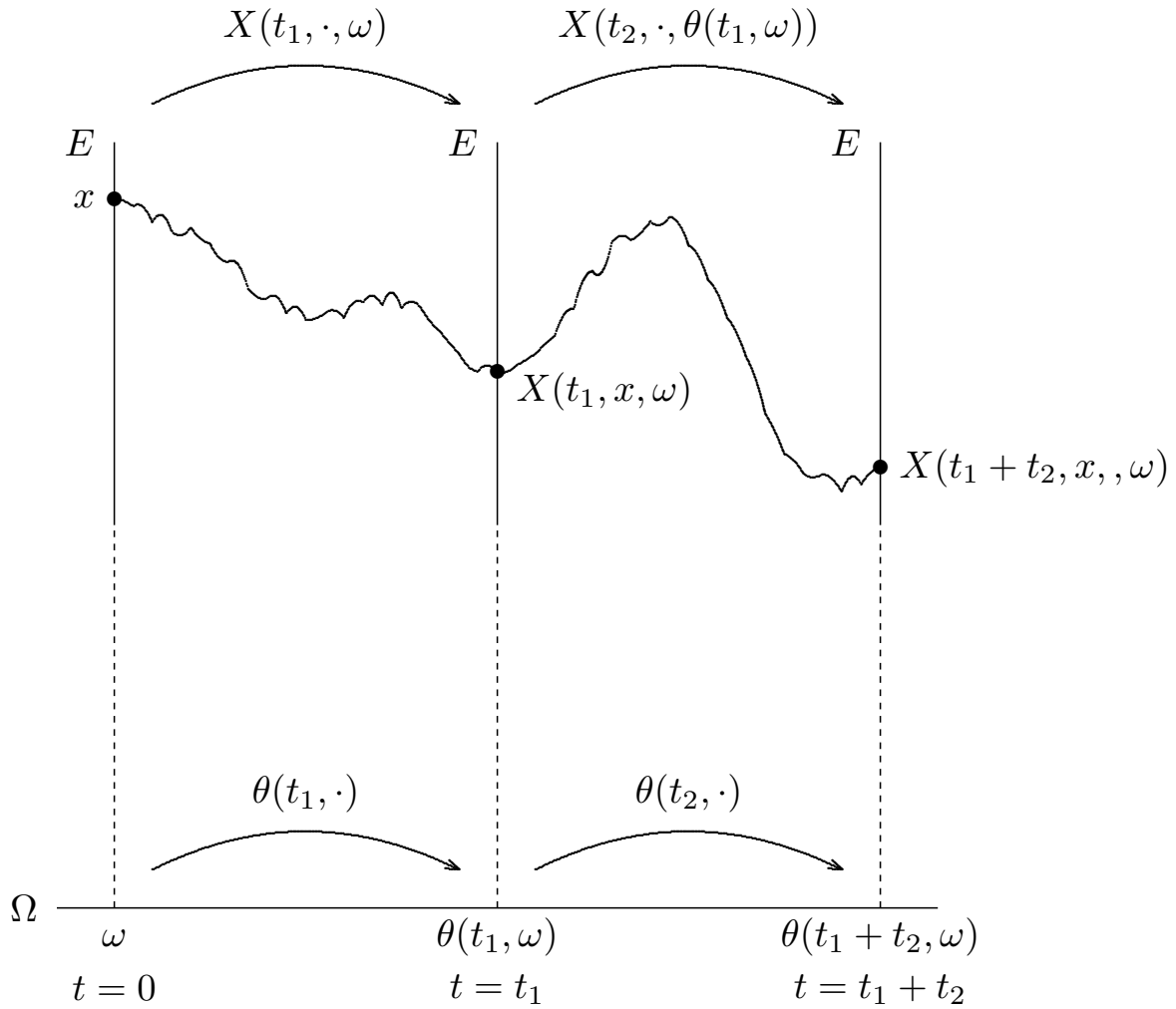
$E :=$ real (separable) Hilbert space, norm $\|\cdot\|$, Borel σ -algebra.

Definition.

$k =$ non-negative integer, $\epsilon \in (0, 1]$. A $C^{k, \epsilon}$ *perfect cocycle* (X, θ) on E is a measurable random field $X : \mathbf{R}^+ \times E \times \Omega \rightarrow E$ such that:

- (i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E$ is continuous; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $E \ni x \mapsto X(t, x, \omega) \in E$ is $C^{k, \epsilon}$ ($D^k X(t, x, \omega)$ is C^ϵ in x).
- (ii) $X(t_1 + t_2, \cdot, \omega) = X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.
- (iii) $X(0, x, \omega) = x$ for all $x \in E, \omega \in \Omega$.

Cocycle Property



Vertical solid lines represent random fibers: copies of E . (X, θ) is a “vector-bundle morphism”.

Definition

A random variable $Y : \Omega \rightarrow E$ is a *stationary point* for the cocycle (X, θ) if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{1}$$

for all $t \in \mathbf{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$.

Linearization. Hyperbolicity.

Linearize a $C^{k,\epsilon}$ cocycle (X, θ) along a stationary random point Y : Get an $L(E)$ -valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of X and chain rule.)

Theorem. (*Oseledec-Ruelle*)

Let $T : \mathbf{R}^+ \times \Omega \rightarrow L(E)$ be strongly measurable, such that (T, θ) is an $L(E)$ -valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(E)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$,

$$\lim_{n \rightarrow \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \dots$$

where the λ_i 's are distinct. Each e^{λ_i} has a fixed finite non-random multiplicity m_i and eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad i > 1.$$

Then

$$\cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i \quad \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega),$$

and

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0$, $i \geq 1$.

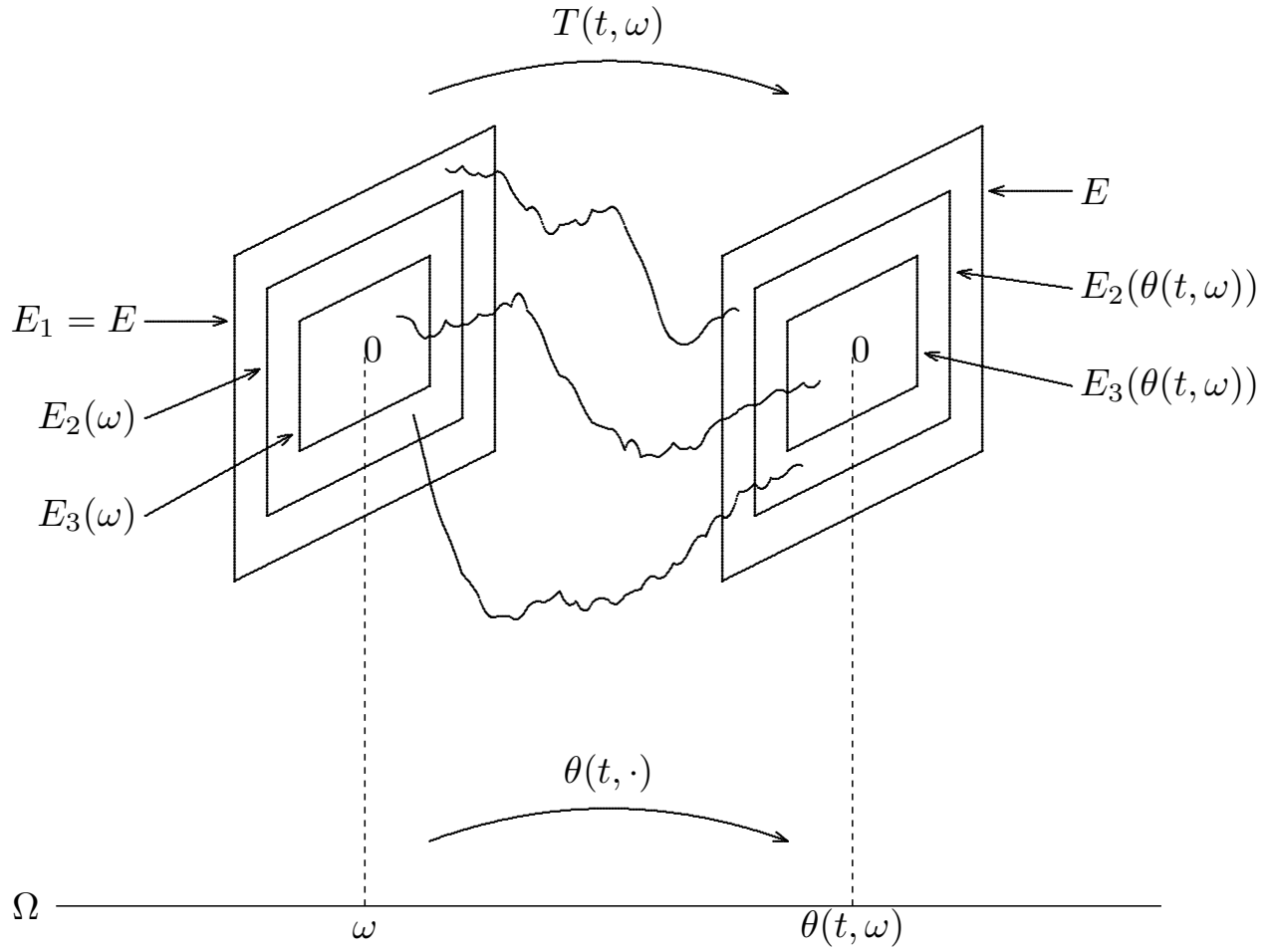
Proof.

Based on discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). \square

Lyapunov Spectrum:

$\{\lambda_1, \lambda_2, \lambda_3, \cdots\} := \text{Lyapunov spectrum of } (T, \theta).$

Spectral Theorem



Definition

A stationary point $Y(\omega)$ of (X, θ) is *hyperbolic* if the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0-1}$.

Suppose

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2 X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)} < \infty.$$

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{x \in E : \lim_{t \rightarrow \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \leq \lambda_i\}, \quad i \geq 1,$$

for all $\omega \in \Omega^*$, a sure event in \mathcal{F} satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle (DX, θ) ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \quad \omega \in \Omega^*,$$

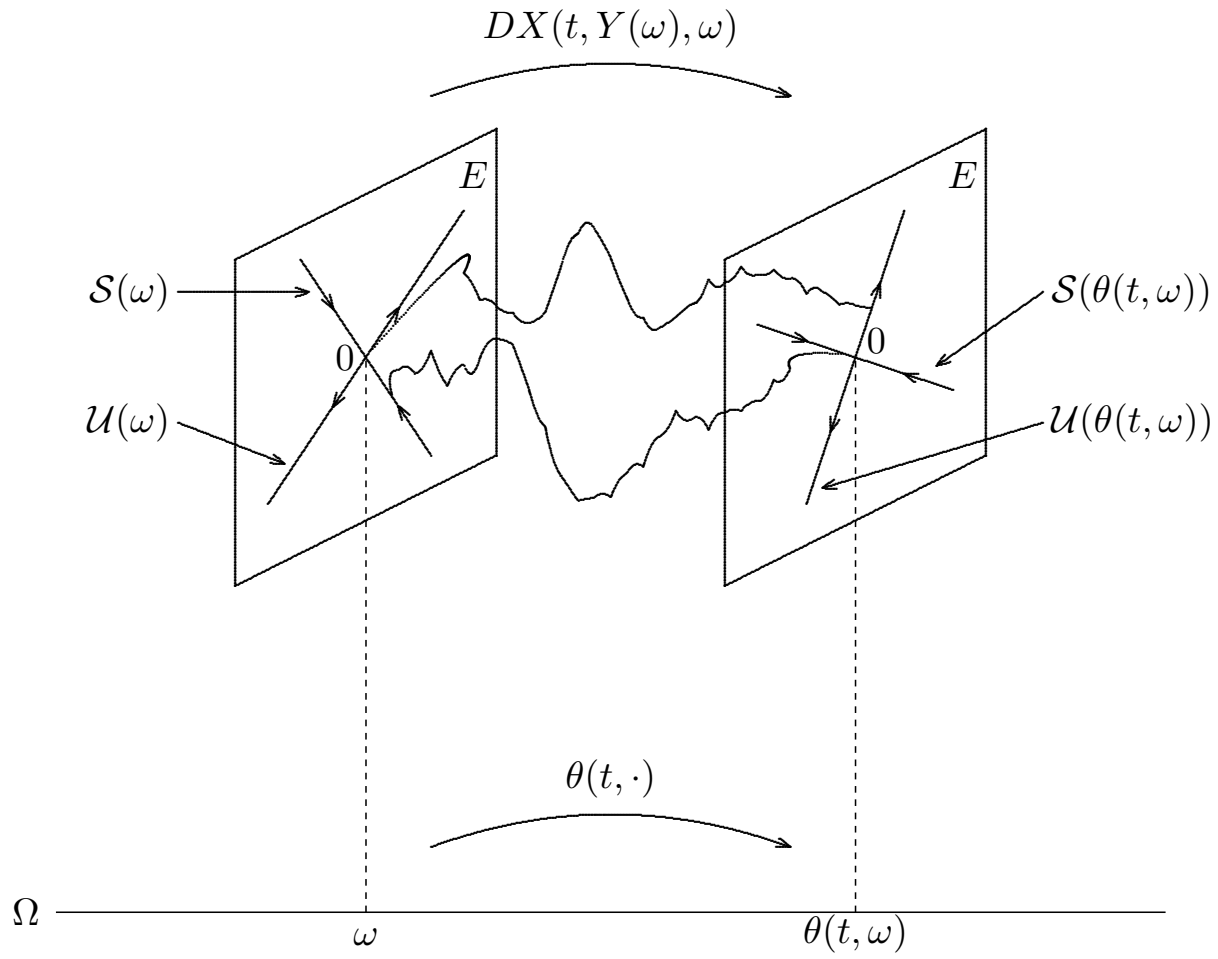
$$DX(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)), \quad DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$, with *exponential dichotomies*

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\| e^{\delta_1 t} \quad \text{for all } t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$

$$\|DX(t, Y(\omega), \omega)(x)\| \leq \|x\| e^{-\delta_2 t} \quad \text{for all } t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

with $\tau_i^* = \tau_i^*(x, \omega) > 0, i = 1, 2$, random times and $\delta_i > 0, i = 1, 2$, fixed.



Stochastic Systems with Memory

“Regular” Itô SFDE with finite memory:

$$\left. \begin{aligned} dx(t) &= H(x(t), x_t) dt + \sum_{i=1}^m G_i(x(t)) dW_i(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (I)$$

Solution segment $x_t(s) := x(t+s)$, $t \geq 0, s \in [-r, 0]$.

m -dimensional Brownian motion $W := (W_1, \dots, W_m)$,
 $W(0) = 0$.

Ergodic Brownian shift θ on Wiener space (Ω, \mathcal{F}, P) .
 $\bar{\mathcal{F}} := P$ -completion of \mathcal{F} .

State space M_2 , Hilbert with usual norm $\|\cdot\|$.

Can allow for “smooth memory” in diffusion coefficient.

$H : M_2 \rightarrow \mathbf{R}^d$, $C^{k,\delta}$, globally bounded.

$G : \mathbf{R}^d \rightarrow L(\mathbf{R}^m, \mathbf{R}^d)$, $C_b^{k+1,\delta}$; $G := (G_1, \dots, G_m)$.

$B((v, \eta), \rho)$ open ball, radius ρ , center $(v, \eta) \in M_2$;

$\bar{B}((v, \eta), \rho)$ closed ball.

Then (I) has a stochastic semiflow $X : \mathbf{R}^+ \times M_2 \times \Omega \rightarrow M_2$
 with $X(t, (v, \eta), \cdot) = (x(t), x_t)$. X is $C^{k,\epsilon}$ for any $\epsilon \in (0, \delta)$, takes

bounded sets into relatively compact sets in M_2 . (X, θ) is a perfect cocycle on M_2 ([M-S.4]).

Theorem. ([M-S], 1999) (*Local Stable and Unstable Manifolds*)

Assume smoothness hypotheses on H and G . Let $Y : \Omega \rightarrow M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\dots < \lambda_{i+1} < \lambda_i < \dots < \lambda_2 < \lambda_1\}$. Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} = -\infty$. (This implies that λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all λ_i are negative, set $\lambda_{i_0-1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

- (i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,
- (ii) $\bar{\mathcal{F}}$ -measurable random variables $\rho_i, \beta_i : \Omega^* \rightarrow (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

- (a) $\tilde{\mathcal{S}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)nr}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0}$$

for all $(v, \eta) \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized semiflow DX is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, $\text{codim } \tilde{\mathcal{S}}(\omega) = \text{codim } \mathcal{S}(\omega)$, is fixed and finite.

$$(b) \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2), (v_1, \eta_1), (v_2, \eta_2) \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$$

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$X(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. Also

$$DX(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

(d) $\tilde{\mathcal{U}}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a unique “history” process $y(\cdot, \omega) : \{-nr : n \geq$

$0\} \rightarrow M_2$ such that $y(0, \omega) = (v, \eta)$ and for each integer $n \geq 1$, one has $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n-1)r, \omega)$ and

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega) e^{-(\lambda_{i_0-1} - \epsilon_2)nr}.$$

Furthermore, for each $(v, \eta) \in \tilde{\mathcal{U}}(\omega)$, there is a unique continuous-time “history” process also denoted by $y(\cdot, \omega) : (-\infty, 0] \rightarrow M_2$ such that $y(0, \omega) = (v, \eta)$, $X(t, y(s, \omega), \theta(s, \omega)) = y(t+s, \omega)$ for all $s \leq 0, 0 \leq t \leq -s$, and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ of the linearized semiflow DX is tangent at $Y(\omega)$ to $\tilde{\mathcal{U}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{\mathcal{U}}(\omega)$ is finite and non-random.

(e) Let $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2$, be the history processes associated with $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{\mathcal{U}}(\omega), i = 1, 2$. Then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \left[\sup \left\{ \frac{\|y(-t, (v_1, \eta_1), \omega) - y(-t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : \right. \right. \\ \left. \left. (v_1, \eta_1) \neq (v_2, \eta_2), (v_i, \eta_i) \in \tilde{\mathcal{U}}(\omega), i = 1, 2 \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\tilde{\mathcal{U}}(\omega) \subseteq X(t, \cdot, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. Also

$$DX(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DX(t, \cdot, \theta(-t, \omega))|_{\mathcal{U}(\theta(-t, \omega))} : \mathcal{U}(\theta(-t, \omega)) \rightarrow \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

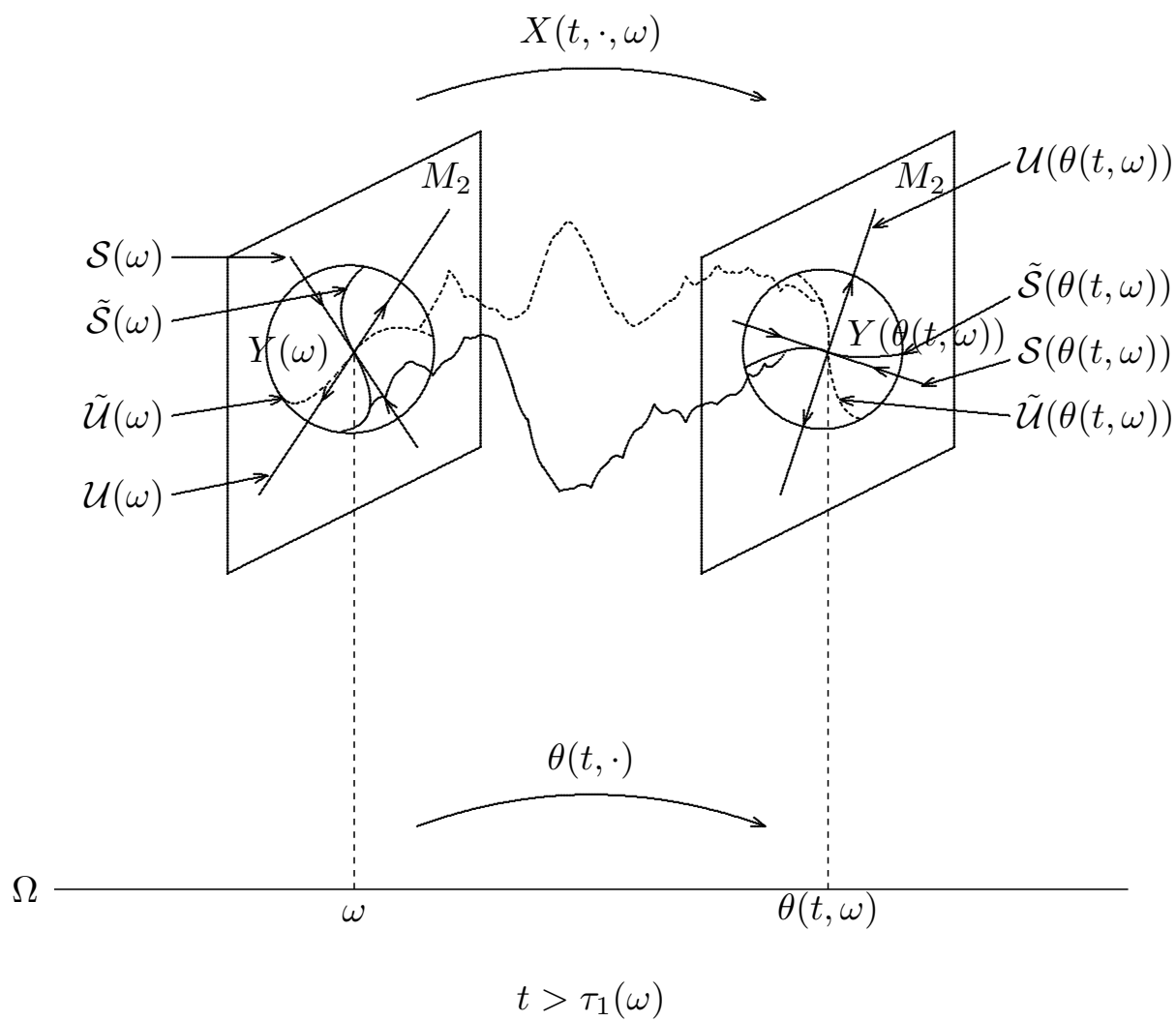
(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ and $\tilde{\mathcal{S}}(\omega)$ are transversal, viz.

$$M_2 = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that H, G are C_b^∞ . Then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞ .

Figure summarizes essential features of Stable Manifold Theorem:

Stable Manifold Theorem



A picture is worth a 1000 words!

Example

Affine linear sfde:

$$\left. \begin{aligned} dx(t) &= H(x(t), x_t) dt + G dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (I'')$$

$H : M_2 \rightarrow \mathbf{R}^d$ continuous linear map, G a fixed $(d \times p)$ -matrix, and W p -dimensional Brownian motion. Assume that the $(d \times d)$ -matrix-valued FDE

$$dy(t) = H \circ (y(t), y_t) dt$$

has a semiflow

$$T_t : L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)) \rightarrow L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)), t \geq 0,$$

which is uniformly asymptotically stable. Set

$$Y := \int_{-\infty}^0 T_{-u}(I, 0) G dW(u) \quad (2)$$

where I is the identity $(d \times d)$ -matrix. Integration by parts and

$$W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbf{R}, \quad (3)$$

imply that Y has a measurable version satisfying (1). Y is Gaussian and thus has finite moments of all orders. See

([Mo.1], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when H is hyperbolic, one can show that a stationary point of (I'') exists ([Mo.1]).

For general white-noise case, get an invariant measure on M_2 for the one-point motion by enlarging probability space. Conversely, let $Y : \Omega \rightarrow M_2$ be a stationary random point independent of the Brownian motion $W(t)$, $t \geq 0$. Then $\rho := P \circ Y^{-1}$ (distribution of Y) is an invariant measure for the one-point motion (by independence of Y and W).

Outline of Proof

- By definition, a *stationary* random point $Y(\omega) \in M_2$ is invariant under the semiflow X ; viz $X(t, Y) = Y(\theta(t, \cdot))$ for all times t .
- Linearize the semiflow X along the stationary point $Y(\omega)$ in M_2 . By stationarity of Y and the cocycle property of X , this gives a linear perfect cocycle $(DX(t, Y), \theta(t, \cdot))$ in $L(M_2)$, where D = spatial (Fréchet) derivatives.
- Ergodicity of θ allows for the notion of *hyperbolicity* of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay r ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is *hyperbolic* if $\lambda_i \neq 0$ for every i .
- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of “perfect versions” of ergodic theorem and Kingman’s sub-additive ergodic theorem. These refined versions give

invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow X .

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle X in a neighborhood of the stationary point Y . Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.
- Introduce the auxiliary perfect cocycle

$$Z(t, \cdot, \omega) := X(t, (\cdot) + Y(\omega), \omega) - Y(\theta(t, \omega)), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr, \cdot, \omega), \theta(nr, \omega))$ near 0 and hence (by translation) for $X(nr, \cdot, \omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t, \cdot)$ -invariant sure event in Ω . This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length r and further refining

the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow X near Y .

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow X . Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for X coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.

REFERENCES

- [A-S] Arnold, L., and Scheutzow, M. K. R., Perfect cocycles through stochastic differential equations, *Probab. Th. Rel. Fields*, 101, (1995), 65-88.
- [C] Carverhill, A., Flows of stochastic dynamical systems: Ergodic theory, *Stochastics*, 14 (1985), 273-317.
- [I-W] Ikeda, N., and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland-Kodansha (1989).
- [Ku] Kunita, H., *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge, New York, Melbourne, Sydney (1990).
- [Mo.1] Mohammed, S.-E. A., The Lyapunov spectrum and stable manifolds for stochastic linear delay equations, *Stochastics and Stochastic Reports*, Vol. 29 (1990), 89-131.
- [Mo.2] Mohammed, S.-E. A., Lyapunov exponents and stochastic flows of linear and affine hereditary Systems, *Diffusion Processes and Related Problems in Analysis*, Vol. II, edited by Mark Pinsky and Volker Wihstutz, Birkhauser (1992), 141-169.
- [M-S.1] Mohammed, S.-E. A., and Scheutzow, M. K. R., Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part I: The multiplicative ergodic theory, *Ann. Inst. Henri*

Poincaré, Probabilités et Statistiques, Vol. 32, 1, (1996), 69-105. pp. 43.

- [M-S.2] Mohammed, S.-E. A., and Scheutzow, M. K. R., Spatial estimates for stochastic flows in Euclidean space, *The Annals of Probability*, 26, 1, (1998), 56-77.
- [M-S.3] Mohammed, S.-E. A., and Scheutzow, M. K. R., The stable manifold theorem for stochastic differential equations, *The Annals of Probability*, Vol. 27, No. 2, (1999), 615-652.
- [M-S.4] Mohammed, S.-E. A., and Scheutzow, M. K. R., The stable manifold theorem for non-linear stochastic hereditary systems. Part I: Existence of the semiflow, Part II: The local stable manifold theorem. (preprints).
- [O] Oseledec, V. I., A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, *Trudy Moskov. Mat. Obšč.* 19 (1968), 179-210. English transl. *Trans. Moscow Math. Soc.* 19 (1968), 197-221.
- [Ru.1] Ruelle, D., Ergodic theory of differentiable dynamical systems, *Publ. Math. Inst. Hautes Etud. Sci.* (1979), 275-306.
- [Ru.2] Ruelle, D., Characteristic exponents and invariant manifolds in Hilbert space, *Annals of Mathematics* 115 (1982), 243-290.
- [Wa] Wanner, T., Linearization of random dynamical systems, in *Dynamics Reported*, vol. 4, edited by U. Kirchgraber and H.O. Walther, Springer (1995), 203-269.